



$$u_{xx} + u_{yy} = 0$$

$$\text{Try } u = X(x)Y(y) \Rightarrow \frac{x''}{x} + \frac{y''}{y} = 0$$

$$\text{If } \frac{x''}{x} = -\pi^2 \text{ so that } x'' + \pi^2 x = 0$$

$$\text{then } \frac{y''}{y} = \pi^2 \Rightarrow y'' - \pi^2 y = 0$$

$$X(0)Y(y) = 0 \Rightarrow X(0) = 0 \text{ or } X'Y + XY' \text{ at } x=L$$

Boundary conditions imply

$$\text{giving } X' + X = 0 \text{ & } X(L)Y'(0) = 0 \Rightarrow Y'(0) = 0 \quad (8)$$

The equation for X has solutions $X = A \sin \pi x$ if in addition

$$\pi A \cos \pi L + A \sin \pi L = 0 \Rightarrow \tan \pi L = -\pi L. \text{ This has an infinite number}$$

of positive roots defining π_n , $\boxed{\tan \pi_n L = -\pi_n L}$

We can assume $\pi_n > 0$ as \sin is odd & the case $\pi_n = 0$ gives $\pi = 0$ $X = A + Bx$ which cannot satisfy the boundary conditions unless $A = B = 0$

The solution for y is $y(y) = A \cosh \pi_n y + B \sinh \pi_n y$ & $y'(0) = 0 \Rightarrow B = 0$.

The general solution is the sum over all possible values of π_n of the

solution X^Y is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin \pi_n x \cosh \pi_n y \quad (\text{similar seen}) \quad (8)$$

$$\text{If } u(h,x) = f(x) \text{ then } f(x) = \sum A_n \sin \pi_n x \cosh \pi_n h \quad \& \quad A_n = \frac{\int_0^L f(x) \sin \pi_n x dx}{\cosh \pi_n h \int_0^L \sin^2 \pi_n x dx}$$

$$\text{At } y=0 \quad \cosh \pi_n y = 1 \quad \& \quad \text{evaluating } \int_0^L \sin^2 \pi_n x dx = \frac{1}{2} \int_0^L 1 - \cos 2\pi_n x dx$$

$$= \frac{1}{2} \left[L - \frac{\sin 2\pi_n L}{2\pi_n} \right] = \frac{1}{2} \left[L - \frac{\sin \pi_n L \cos \pi_n L}{\pi_n} \right] \quad \text{but } \sin \pi_n L = -\pi_n L \cos \pi_n L$$

$$= \frac{L}{2} \left[1 + \cos^2 \pi_n L \right] \quad \text{giving result.}$$

(this unseen)

(9)

$$2) \quad x^2 y'' + (x^2 - x)y' + (1+cx)y = 0 \quad \text{is} \quad y'' + \left(1 - \frac{1}{x}\right)y' + \left(\frac{1}{x^2} + \frac{1}{x}\right)y$$

pole of order 1 pole of order 2
 ⇒ regular singular point

If $y = \sum_{n=0}^{\infty} a_n x^{n+c}$ then $\sum_{n=0}^{\infty} \left[(n+c)(n+c-1) - (n+c) + 1 \right] a_n x^{n+c} + \sum_{n=0}^{\infty} [(n+c)+1] a_n x^{n+c+1} = 0$

Initial eqn is $c(c-1) - c + 1 = (c-1)^2 = 0 \Rightarrow c=1$ repeated.

Coefficient of x^{n+c} is 0 $\Rightarrow a_n = -\frac{[n+c+1] a_{n-1}}{(n+c-1)^2} = -\frac{a_{n-1}(n+c)}{(n+c-1)^2}$

Repeated use, with $a_0=1$ gives

$$a_n = (-1)^k \frac{(k+c)(k+c-1) \dots (2+c)(1+c)}{(k+c-1)^2 (k+c-2)^2 \dots (1+c)^2 c^2} = \frac{(-1)^k (k+c)}{(k+c-1)(k+c-2) \dots (1+c)c} \quad (i)$$

Putting $c=1$ gives $a_n = (-1)^k \frac{(k+1)}{k!}$

The second soln is $y_2 = \ln x y_1 + \sum_{n=1}^{\infty} \frac{\partial a_n}{\partial c} \Big|_{c=1} x^{k+1}$

Now $\ln a_n = \ln(-1)^k + \ln(n+c) + \sum_{r=0}^{k-1} (-1) \ln(n+r+c) - \ln c$

$$\Rightarrow \frac{\partial a_n}{\partial c} = a_n \left\{ \frac{1}{n+c} - \sum_{r=0}^{k-1} \frac{1}{n+r+c} - \frac{1}{c} \right\}$$

& at $c=1$ this is $(-1)^k \frac{(k+1)}{k!} \left\{ \frac{1}{k+1} - \sum_{r=1}^k \frac{1}{r} - 1 \right\} = (-1)^k \frac{(k+1)}{k!} (-1) \left\{ \frac{k}{k+1} + \sum_{r=1}^k \frac{1}{r} \right\}$

so

$$y_2 = x \ln x \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)x^k}{k!} - x \sum_{k=0}^{\infty} x^k (-1)^k \frac{(k+1)}{k!} \left[\frac{k}{k+1} + \sum_{r=1}^k \frac{1}{r} \right] \quad (ii)$$

similar seen)

$$3) \text{ a) } G(\omega, x) = \frac{1}{(1-2\omega x + \omega^2)^{1/2}} = \sum_{n=0}^{\infty} \omega^n P_n$$

$$\frac{\partial G}{\partial \omega} = \frac{x-\omega}{(1-2\omega x + \omega^2)^{3/2}} \Rightarrow (1-2\omega x + \omega^2) \frac{\partial G}{\partial \omega} = (x-\omega) G$$

$$\Rightarrow (1-2\omega x + \omega^2) \sum_{n=0}^{\infty} \omega^n P_n = (x-\omega) \sum_{n=0}^{\infty} \omega^n P_n$$

(coeff ω^n)

$$\Rightarrow (n+1)P_{n+1} - 2nP_n + (n-1)P_{n-1} = xP_n - P_{n-1}$$

$$\Rightarrow (n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1} \quad (\text{seen}) \quad (10)$$

b) Multiply both sides of G by themselves & integrate w.r.t x

$$\int_{-1}^1 \frac{1}{1-2\omega x + \omega^2} dx = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \omega^{n+m} \int_{-1}^1 P_n P_m dx = \sum_{n=0}^{\infty} \omega^{2n} \int_{-1}^1 P_n^2 dx$$

but LHS is $\left[-\frac{1}{2\omega} \ln(1-2\omega x + \omega^2) \right]_{-1}^1 = -\frac{1}{2\omega} \ln(1-\omega)^2 + \frac{1}{2\omega} \ln(1+\omega)^2$

$$= \frac{1}{\omega} [\ln(1+\omega) - \ln(1-\omega)] \quad (10)$$

Looking for the coefficient of ω^{2n} . $\sum_{n=0}^{\infty} \omega^n \frac{(-1)^{n+1}}{n} \text{ & } \ln(1-\omega) = -\sum_{n=1}^{\infty} \omega^n / n$

$$\ln(1+\omega) = \omega - \omega^2/2 + \omega^3/3 - \dots = \sum_{n=1}^{\infty} \omega^n \frac{(-1)^{n+1}}{n}$$

so the L.H.S is $+2 \sum_{n=0}^{\infty} \frac{\omega^{2n}}{2n+1} = \sum_{n=0}^{\infty} \omega^n \int_{-1}^1 P_n^2 dx$

$$\Rightarrow \int_{-1}^1 P_n^2 dx = \frac{2}{2n+1} \quad (5)$$

(seen as optional
h.w. question)

$$\begin{aligned}
 \text{a) } \widehat{f * g} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{\infty} f(x-y)g(y)dy dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} e^{-ikx} f(x-y)dx dy \\
 &\stackrel{x-y=u}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} e^{-ik(y+u)} f(u) du dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) e^{-iky} dy \int_{-\infty}^{\infty} f(u) e^{-iku} du \\
 &= \sqrt{2\pi} \widehat{f} \widehat{g} \quad (\text{seen})
 \end{aligned}$$

(10)

$$\begin{aligned}
 \text{b) If } g(\alpha, x) &= \frac{1}{\alpha^2 + x^2}, \text{ then } \widehat{g(\alpha, x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\alpha^2 + x^2} dx = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\cosh t}{\alpha^2 + t^2} dt \\
 &= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2\alpha} e^{-\alpha|k|} = \frac{\sqrt{\pi}}{2} \frac{e^{-\alpha|k|}}{\alpha}
 \end{aligned}$$

The equation is $h * g(\alpha, x) = g(b, x)$ & taking transforms gives, using above $\widehat{h * g} = \sqrt{2\pi} \widehat{f} \widehat{g}(\alpha, x) = \widehat{g}(b, x) \Rightarrow \widehat{h} = \frac{1}{\sqrt{2\pi}} \frac{e^{-bk}}{b} \cdot \frac{\alpha}{e^{-\alpha|k|}} = \frac{1}{\sqrt{2\pi}} \frac{\alpha}{b} e^{-(b-a)|k|}$

So $\widehat{h} \in \mathbb{C}$ & From above $\widehat{f} \left(\frac{1}{e^{-\alpha|k|}} \right) = \sqrt{\frac{2}{\pi}} \alpha \frac{1}{\alpha^2 + x^2}$

$$\Rightarrow h(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{\alpha}{b} \cdot \sqrt{\frac{2}{\pi}} \frac{(b-a)}{(b-a)^2 + x^2} = \frac{1}{\pi} \frac{\alpha}{b} \frac{(b-a)}{(b-a)^2 + x^2} \quad (\text{similar seen}) \quad (15)$$

5) $u_t = u_{xx} \Rightarrow \hat{u}_t = -k^2 \hat{u} \Rightarrow \hat{u} = A e^{-k^2 t}$ & if $u(x,0) = f$
 Then $A = \hat{f}$.

Using the convolution result $\hat{f} * \hat{g} = \sqrt{2\pi} \hat{f} \hat{g}$ gives $u(x,t) = f * h = \int_{-\infty}^{\infty} f(x-q) h(q) dq$

where $\sqrt{2\pi} \hat{h} = e^{-k^2 t}$

$$\text{Therefore } h(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-k^2 t} dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t(k^2 - ikx)} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t(k - i \frac{x}{2t})^2 + t(-x^2/4t^2)} dk$$

$$= \frac{1}{2\pi} e^{-x^2/4t} \cdot \sqrt{\frac{\pi}{t}}, \text{ using given result with } -ix/2t = b$$

$$\Rightarrow h(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \quad (\text{seen}) \quad (13)$$

$$\text{Also } u(x,t) = \int_{-\infty}^{\infty} e^{-(x-q)^2} \cdot e^{-q^2/4t} dq / \sqrt{\pi t} = \int_{-\infty}^{\infty} e^{-q^2 \left(\frac{1+4t}{4t}\right) + 2qx - x^2} \frac{dq}{\sqrt{\pi t}}$$

$$= \frac{1}{\sqrt{1+4t}} \int_{-\infty}^{\infty} e^{-\frac{(1+4t)}{4t} \{ q + \varphi x \}^2} e^{-x^2 \{ 1 - \varphi^2(1+4t)/4t \}} dq$$

where $-2\varphi(1+4t)/4t = 2$

$$= \frac{1}{\sqrt{1+4t}} \frac{\sqrt{\pi}}{\sqrt{1+4t}} \sqrt{1+4t} \cdot e^{-x^2/(1+\varphi)}, \quad 1+\varphi = 1 - \frac{4t}{1+4t} = \frac{1}{1+4t}$$

$$= \frac{1}{\sqrt{1+4t}} e^{-x^2/(1+4t)}$$

(unseen) (12)

$$6) \text{ a) i) } L[e^{-at}] = \int_0^\infty e^{-st} e^{-at} dt = \int_0^\infty e^{-(s+a)t} dt = \frac{1}{s+a}$$

$$\text{ii) } L\left[\frac{df}{dt}\right] = \int_0^\infty e^{-st} \frac{df}{dt} dt = \left[e^{-st} f \right]_0^\infty + s \int_0^\infty e^{-st} f dt \\ = -f(0) + s \bar{f}. \quad (\text{see } 1) \quad (5)$$

$$\text{b) } u_t + x u_x = x^n, \quad u(x,0) = x^m, \quad u(0,t) = 0$$

Take transforms $s\bar{u} - u(x_0) + x\bar{u}_x = x^n \bar{t} = x^n/s$

$$\Rightarrow x\bar{u}_x + s\bar{u} = x^n/s + x^m \Rightarrow \bar{u}_x + \frac{s}{x}\bar{u} = \frac{x^{n-1}}{s} + \frac{x^{m-1}}{x}$$

Integrating factor is $x^s \Rightarrow \frac{d}{dx} \left[x^s \bar{u} \right] = \frac{1}{s} x^{n+s-1} + x^{m+s-1}$

$$\Rightarrow x^s \bar{u} = \frac{1}{s(n+s)} x^{n+s} + \frac{1}{m+s} x^{m+s} + A \Rightarrow \bar{u} = \frac{x^n}{s(n+s)} + \frac{x^m}{m+s} + \frac{A}{x^s} \quad (11)$$

$$u(0,t) = 0 \Rightarrow \bar{u}(0,s) = 0 \Rightarrow A = 0$$

Now invert $\frac{1}{s(n+s)} = \frac{1}{n} \left[\frac{1}{s} - \frac{1}{n+s} \right]$ & using result above

$$u(x,t) = \frac{x^n}{n} \left[1 - e^{-nt} \right] + x^m e^{-mt} \quad (\text{similar seen})$$